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From Photon to Neuron Chapter 15: Planetary Climates

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From Photon to Neuron Chapter 15: Planetary Climates

Abstract

This chapter extends Part III of the book *From Photon to Neuron* (Princeton Univ Press 2017). This preliminary version is made freely available as-is in the hope that it will be useful.

Disciplines

Physical Sciences and Mathematics | Physics

CHAPTER 15

Planetary Climates

Dazzling and tremendous how quick the sun-rise would kill
me,

If I could not now and always send sun-rise out of me.

— *Walt Whitman*

15.1 SIGNPOST: *TRANSFORMATION*

Armed with our understanding of the way that atoms and molecules emit and absorb light, we can now say some things about our planet and its neighbors. Is this biological physics? It is true that the Earth system, or another planet, does not reproduce and hence did not evolve via descent with modification in the presence of natural selection. In that sense, planets are not “alive.” But planetary systems are complex and self-regulating, with many interesting feedbacks, so the subject has much in common with the physiology of living organisms. Moreover, our own planet hosts a biosphere that has itself made major modifications to the climate. So whether or not we call it “alive,” some of the concepts we have developed turn out to be relevant.

This chapter will start by revisiting the thermal radiation spectrum, then gradually construct a model that can help us understand the broad outlines of planetary climates. Besides being an intellectual challenge, understanding our own planet’s climate, and our influence on it, is an urgent current problem. It is remarkable that quantum physics, usually thought of as limited to the world of very tiny things, plays two decisive roles in this story. We will see how the properties of thermal radiation, in conjunction with the *transformation* of energy by a planet’s surface and atmosphere, control its overall climate.¹

15.2 THERMAL RADIATION WITHOUT BLACK BODIES

15.2.1 Closed chamber

Section 1.3.3’c (page 52) obtained the spectral energy density of light in thermal equilibrium:

$$u_\nu(\nu)d\nu = \frac{16\pi^2\hbar\nu^3}{c^3} (e^{2\pi\hbar\nu/(k_B T)} - 1)^{-1} d\nu. \quad [1.19, \text{page 52}]$$

Equation 1.19 is a very specific family of functions, with only a single free parameter (the absolute temperature T). Thus, for example, the location of the peak has a definite relationship to the magnitude of u_ν at that peak. However, Chapter 1 said little about *how* light can come to thermal equilibrium, nor even what that phrase might mean.

¹William Berner, Craig Bohren, Kevin Chen, and Philip Sieg contributed to this chapter.

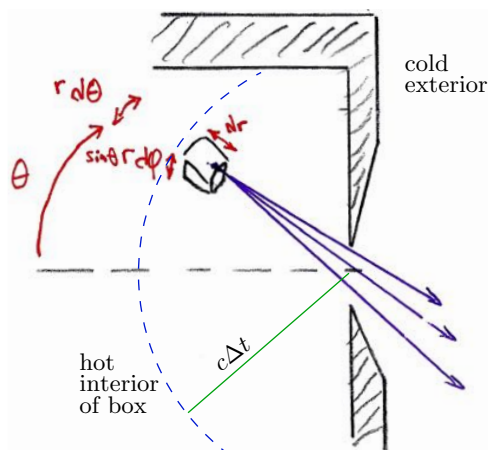


Figure 15.1: [Sketch.] **Connection between energy density and outgoing flux** of thermal radiation. A small volume at polar coordinate location (r, θ, ϕ) contains photons, some of which can stream through a small opening.

Consider first a closed chamber with nothing (vacuum) inside it. Some agency keeps the walls of the chamber at a fixed, uniform temperature T . The walls consist of atoms, which have random motions determined by the value of T (Section 0.6, page 17). Collisions between atoms can convert some of the energy of this thermal motion into atomic state changes, which then can lead to photon emission. Conversely, photon absorption can give an atom energy that eventually finds its way back into thermal motion.²

The continual exchange of energy between walls and photons leads to an equilibrium state. If we later increase the wall temperature, then emission outstrips absorption, depositing energy into the photons until they arrive at a new spectrum, again of the form Equation 1.19 but with the new value of T . We suppose that the walls are so thick and dense that no light escapes the chamber, at least in some range of frequencies. Then at least in that range, we expect to find photons in the cavity with the thermal spectrum (Equation 1.19).

15.2.2 Almost-closed chamber

Suppose now that we make a small opening in the wall of the chamber, with area $d^2\Sigma$. Photons can now stream freely out of that opening. Suppose that the exterior space is dark; perhaps the chamber is insulated and the outer world is cold. If the opening is small compared to the overall size of the chamber, then the loss of a few photons will not affect the interior conditions much, and we can expect that the spectral energy density inside the chamber, right up to the opening, will still be given by Equation 1.19.

Consider a small volume $d^3\mathbf{r}$ inside the chamber, centered at a position \mathbf{r} relative to the center of the opening (Figure 15.1). A photon inside this volume can only escape through the opening in time dt if the distance to the opening is smaller than $c\Delta t$. Thus, only a hemispherical region can contribute (dotted line in Figure 15.1). Moreover, the outgoing photon must be aimed at the exit hole; that is, its direction must fall within the angular area $\cos\theta d^2\Sigma/r^2$ subtended at \mathbf{r} by the opening.³ Here θ

²Although an isolated atom at rest can only emit and absorb at discrete frequencies, in dense solid matter the list of allowed frequencies merges into continuous bands.

³See Section 6.7.2. To understand the θ dependence, suppose that the opening lies parallel to \mathbf{r} . Then $\cos\theta = 0$; indeed, no photons from this volume element can exit through such an opening.

is the angle that \mathbf{r} makes with the perpendicular, so $\cos \theta$ is the foreshortening from viewing the opening at this angle. The photons near \mathbf{r} are Uniformly distributed in angle, so a fraction $\cos \theta d^2 \Sigma / (4\pi r^2)$ are eligible to escape, carrying their energy out of the box.

Using spherical polar coordinates, we can write the energy escaping in time Δt and frequency range $d\nu$ as an integral over $d^3 \mathbf{r}$ of the spectral energy density $u_\nu d\nu$ times the fraction that escape:

$$\int_0^{c\Delta t} r^2 dr \int_0^1 d(\cos \theta) \int_0^{2\pi} d\phi u_\nu d\nu \frac{\cos \theta d^2 \Sigma}{4\pi r^2}.$$

We can simplify this expression by letting $\mu = \cos \theta$. Because u_ν is independent of direction, we find

$$u_\nu d\nu \frac{d^2 \Sigma}{4\pi} 2\pi \int_0^1 \mu d\mu \int_0^{c\Delta t} \frac{r^2 dr}{r^2} = \frac{1}{2} u_\nu d\nu d^2 \Sigma \frac{1}{2} c \Delta t. \quad (15.1)$$

This quantity is often written as $B_\nu(\nu; T) d\nu d^2 \Sigma \Delta t$, where B_ν is called the **Planck spectral irradiance function**. Equations 1.19 and 15.1 give it as

$$B_\nu(\nu; T) = \frac{4\pi^2 \hbar \nu^3}{c^2} (e^{2\pi \hbar \nu / (k_B T)} - 1)^{-1}. \quad (15.2)$$

The following table summarizes various forms of the Planck spectrum:⁴

term	description	formula
spectral energy density	energy per volume per frequency interval	$u_\nu = (16\pi^2 \hbar \nu^3 / c^3) / (e^x - 1)$
spectral photon density	photons per volume per frequency interval	$u_\nu / (2\pi \hbar \nu)$
photon PDF	probability per frequency interval	$\wp_\nu = u_\nu / \left(\nu \int d\nu' u_\nu(\nu') / \nu' \right)$
spectral irradiance	power per area per frequency interval	$B_\nu = u_\nu c / 4$
spectral photon arrival rate	photons per time per frequency interval	$B_\nu \Sigma / (2\pi \hbar \nu)$
spectral energy density	energy per volume per wavelength interval	$u_\lambda = (16\pi^2 \hbar c / \lambda^5) / (e^x - 1)$
spectral photon density	photons per volume per wavelength interval	$u_\lambda \lambda / (2\pi \hbar c)$
photon PDF	probability per wavelength interval	$\wp_\lambda = u_\lambda \lambda / \int d\nu' u_\lambda(\lambda') \lambda'$
spectral irradiance	power per area per wavelength interval	$B_\lambda = u_\lambda c / 4$
spectral photon arrival rate	photons per time per wavelength interval	$B_\lambda \lambda \Sigma / (2\pi \hbar c)$

Various forms of the Planck formula for the spectrum of thermal radiation. In these formulas, $x = 2\pi \hbar \nu / (k_B T) = 2\pi \hbar c / (\lambda k_B T)$ and Σ is the area of a patch of emitting surface.

15.2.3 Isolated radiating body

A red-hot horseshoe pulled from the forge gives off light, but it may not seem at all analogous to the sort of situation envisioned so far. Nevertheless, inside this lump of iron there are lots of photons interacting strongly with lots of atoms, so we may hope that they will have a nearly equilibrium spectrum. Only the photons very near

⁴Section 3.5'a (page 133) introduced irradiance, which has typical units W m^{-2} . The *spectral* irradiance has units $\text{W m}^{-2} \text{Hz}^{-1}$ and is a function of frequency (Section 3.5'a, page 133). It can be converted to spectral photon flux irradiance by dividing by the energy of a photon with frequency ν .

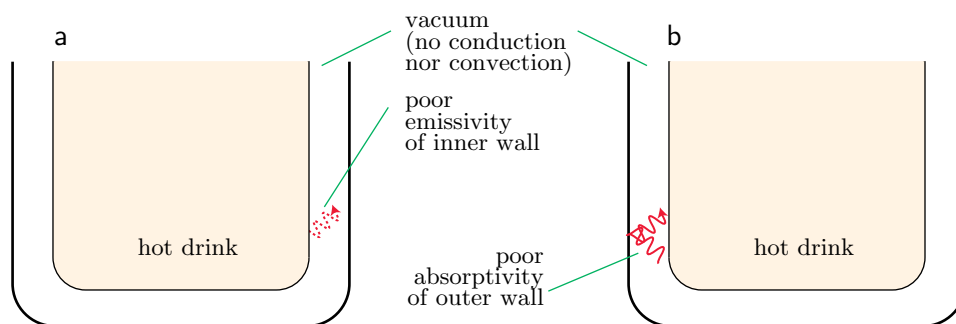


Figure 15.2: [Sketch.] **A vacuum flask.** (a) There is no material substance between the walls of the flask to conduct thermal energy, and the silvered inner wall does not readily emit IR photons, so energy is trapped inside. (b) The silvered outer wall of the flask does not readily absorb IR photons, so any energy that does escape from the inner wall is mainly reflected instead of heating the outer wall.

to the surface are able to exit without suffering further interactions, so even the unobstructed surface of the body may act as a “small window” in the sense of the previous paragraphs. Indeed, hot lumps of iron, incandescent light bulb filaments, and many other such macroscopic objects do emit photons at a rate proportional to the surface area and with a spectrum close to that of thermal radiation, with T equal to the temperature of the object. Even humans emit thermal radiation, with $T \approx 310$ K.

The surface of a material body is not, of course, really a small open hole, and so it may further impede the release of photons. For this reason, any such body has an empirically determined factor called its **emissivity** function, $\mathcal{E}(\nu)$. The emissivity at a given frequency is the ratio of the object’s actual emitted spectral irradiance to the Planck function $B_\nu(\nu; T)$ evaluated at the object’s temperature. The usefulness of this definition stems from the fact that, for many bodies, emissivity is often roughly constant over an interesting range of frequencies and temperatures.

Although the Sun is not solid matter, it has an outermost layer that interacts strongly with light and that has a well-defined temperature. Thus, that layer is effectively an “object” with nearly 100% emissivity over a broad frequency range.⁵ Before it enters our atmosphere, light from the Sun is indeed measured to be nearly of Planck form, with spectrum given by Equation 15.2 with $T \approx 5800$ K. In particular, both the location of the peak and value of the spectral irradiance at that peak are reasonably well fit with a single choice for the value of T .

Many other objects also have IR emissivity approaching 100%, but there are noteworthy exceptions: For example, the silvered inner glass wall of a vacuum flask has been engineered to emit very little infrared into the vacuum cavity, keeping its contents hot (Figure 15.2a).

15.2.4 Optically thin body

One reason the Sun interacts strongly with light is that its atoms have been ionized (the electrons move independently of the nuclei). Ordinary gases consist of intact (neutral) atoms, and hence interact much less strongly. Thus, a sample of gas (or a particular atmospheric zone) may have enough molecular collisions to give it a well

⁵Even a gas will have some broadening of its spectral lines, due to molecular collisions, Doppler shifts, and interactions between many finely-spaced energy levels, so we again expect to find bands in which the spectrum of light takes the thermal form.

defined temperature, and yet not enough photon interactions to bring its light emission to Planck form. It may nevertheless be possible to describe such an “optically thin” sample simply by assigning it an appropriate emissivity value smaller than 100%.

15.2.5 Total power output

The total power output from a radiating body is the integral of its spectral irradiance over all frequencies and over all of its surface. If the body’s spectrum is of Planck form, then the frequency integral can be computed by using the fact that

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}.$$

Thus the total irradiance is⁶

$$\text{irradiance} = \sigma_{\text{SB}} T^4, \quad \text{\textbf{Stefan–Boltzmann law}} \quad (15.3)$$

where the **Stefan–Boltzmann constant** is

$$\sigma_{\text{SB}} = \frac{\pi^2 k_{\text{B}}^4}{60 \hbar^3 c^2} \approx 5.7 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}.$$

If the body’s emissivity is smaller than 100%, but is roughly constant over the relevant frequency range, then Equation 15.3 can simply be reduced by that factor.

Your Turn 15A

Show that the T^4 dependence in Equation 15.3 can be predicted solely on the basis of dimensional analysis.

15.2.6 Absorptivity

The emission of thermal radiation comes at the expense of random molecular motion (sometimes called thermal motion). Conversely, an illuminated object will transform at least some of the energy that lands on it into thermal form. For example, consider again the almost-closed chamber in Figure 15.1. Any photon that wanders *into* the small hole from the outside will interact with the interior walls. Its energy will be added to the general store of thermal energy in the walls, so all of the incoming photons are eventually absorbed; we say the hole itself is “black.” More generally, at any frequency a body will absorb some fraction of incoming photons and transfer their energy to thermal motion. That fraction is called the body’s “absorptivity” at that frequency. The rest of the incoming photons may be scattered (for example, by reflection) or transmitted (as visible light is transmitted by glass). But even ordinary window glass is highly absorbing at non-visible frequencies, including infrared.

Remarkably, emissivity and absorptivity are related in a simple way: They are always equal at any frequency, a result called **Kirchhoff’s law**. Thus, special “low emissivity glass” for energy-efficient buildings reflects incident infrared light instead

⁶Figure B.1 shows some experimental data, including the very rapid rise of energy output with increasing temperature.

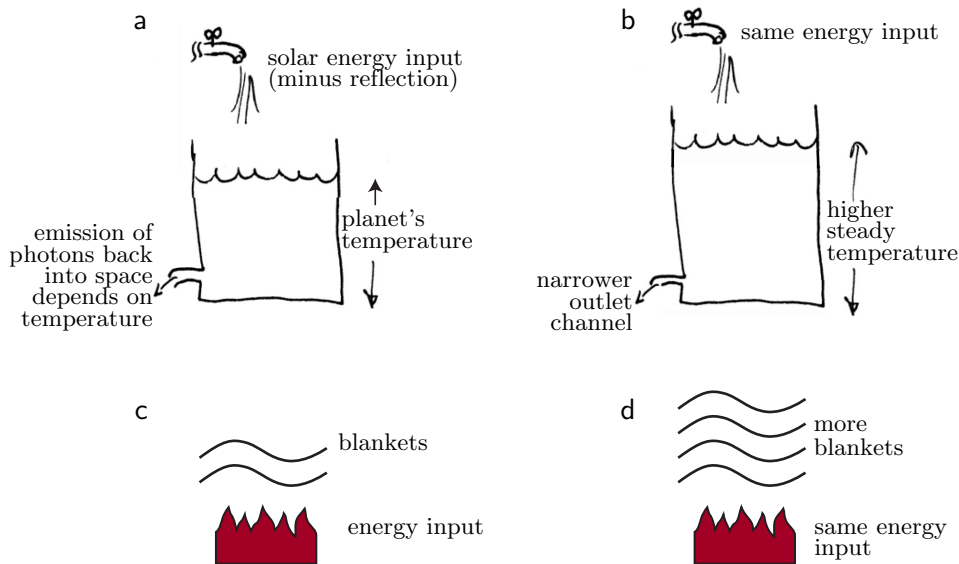


Figure 15.3: [Metaphor.] **Two self-regulating systems.** (a) Analogy with fluids. The escape rate depends on the water level. The level rises until the escape rate equals the inflow, then becomes steady. (b) Impeding water exit without changing the inflow shifts the system to a new, higher steady level. (c) Adding energy at a fixed rate and impeding its loss with blankets leads to a steady temperature. (d) Adding more blankets leads to a higher steady-state temperature for the same energy input rate.

of absorbing it.⁷ And the outer silvered wall of a vacuum flask, which is physically identical to the inner wall, relies on its low absorptivity to bounce any infrared light that made it out of the inner wall without itself heating up (Figure 15.2b). Conversely, that same vacuum flask is equally good at *preventing* energy from *entering* and warming a *cold* drink.

15.3 MARS AND THE MOON

We can now begin to think about the temperature of a rock, for example, a planet such as Earth, Venus, or Mars, that is heated by a distant source of light. Each of these planets has a light and a dark side at any moment, but each is rotating, so we will make the rough approximation that each is subjected to a solar energy infall that is the average of day and night. Each is of course spherical, so incoming light lands at various angles depending on position on the surface: Generally the angle is more oblique at the poles than at the equator. We will make the even rougher approximation of neglecting this difference, imagining that all energy from the Sun is spread uniformly over the planet's surface.

Figure 15.3a shows a system with a simple self-regulation mechanism. Water flows into a bucket at a fixed rate. A leak at the bottom of the bucket lets water out at a rate that depends on its pressure. The pressure increases with increasing depth of water in the bucket. Suppose that the bucket starts empty. Initially, inflow outstrips

⁷Ordinary glass has infrared emissivity, and hence absorptivity, around 90%. Thus, it freely absorbs IR, then radiates that energy in both directions. The resulting transport of energy is undesirable both in summer and in winter; coated glass with ϵ as low as 2% can reduce that transport sharply.

outflow, and the level rises. Eventually, however, the system arrives at a **steady state**, in which its water level sets an exit rate that matches the inflow. If later we increase the inflow rate, the system will stabilize at a new, higher steady level. If instead we shrink the exit hole, we again find that the eventual steady level is higher than initially (panel (b)).

Similarly, we can think of the blankets on a bed as impeding the flow of energy from a fixed source (you). Adding more blankets is analogous to decreasing the leak in the hydraulic analogy (Figure 15.3d). A fixed radiant energy input to a planet's surface also brings it a steady state in which its output of thermal radiation balances the input. To find how much energy arrives at the planet surface, note that each square meter of the Sun gives off energy at a rate governed by the Stefan–Boltzmann law, a total of $4\pi R_{\text{sun}}^2 \sigma_{\text{SB}} T_{\text{sun}}^4$, where $R_{\text{sun}} \approx 7.0 \times 10^8 \text{ m}$. Most of this energy disappears into space, missing the planet. Only the fraction $\pi R_p^2 / (4\pi D_p^2)$ intercepts the planet, where R_p is the planet's radius and D_p its distance. Here are some rough numbers:⁸

planet	radius R_p [10^6 m]	distance D_p [10^{11} m]	reflectivity \mathcal{R}	observed mean surface temperature T_{surf} [K]
Earth	6.4	1.50	0.31	288
Venus	6.1	1.08	0.77	737
Mars	3.4	2.3	0.25	210

Some of the incoming solar energy is reflected, for example, by snow. We account for this by reducing the solar input by a factor $1 - \mathcal{R}$, where the constant \mathcal{R} is a dimensionless number determined by observation.⁹ This reduced total input is called the planet's **net insolation**.¹⁰ Each planet (and even some moons) also has internal energy production (geothermal energy), but this source turns out to be negligibly small for the three we have chosen to study. So the mean rate of energy input per planet surface area is given by

$$(1 - \mathcal{R})\sigma_{\text{SB}}T_{\text{sun}}^4 R_{\text{sun}}^2 / (4D_p^2). \quad (15.4)$$

In short, the Sun's thermal radiation output is reduced by reflection and the geometrical factors in Equation 15.4.

The incoming photons are absorbed (we have already accounted for reflection). Absorbing energy at this huge rate should continually warm the planet, yet planetary temperatures are roughly stable. For that observation to be compatible with the conservation of energy, it must be the case that *every watt of insolation gets sent back into space* from the planet.

For a planet with little or no atmosphere (a “rock” or “naked planet”), there is only one way to shed energy to outer space: The surface temperature rises until its thermal emission, $\mathcal{E}\sigma_{\text{SB}}T_{\text{surf}}^4$ matches the net insolation. Because the incoming energy flow is reduced by the factors in Equation 15.4, however, we do not expect $T_{\text{surf}} = T_{\text{sun}}$. Instead, the predicted surface temperature T_{surf} is far lower than T_{sun} (Figure 15.4).

⁸The table lists distance values computed simply as the average of the closest and farthest points on each planet's orbit.

⁹In the case of Earth and Venus there are also reflective clouds, aerosols, and so on. Although we are not yet including the effects of atmosphere, we have incorporated cloud reflection in the data of the table via the \mathcal{R} values. Specifically, \mathcal{R} is the reflectivity averaged over the frequency range typical of the solar spectrum, sometimes called “Bond albedo.”

¹⁰“Insolation” refers to the total incoming solar energy per planetary surface area.

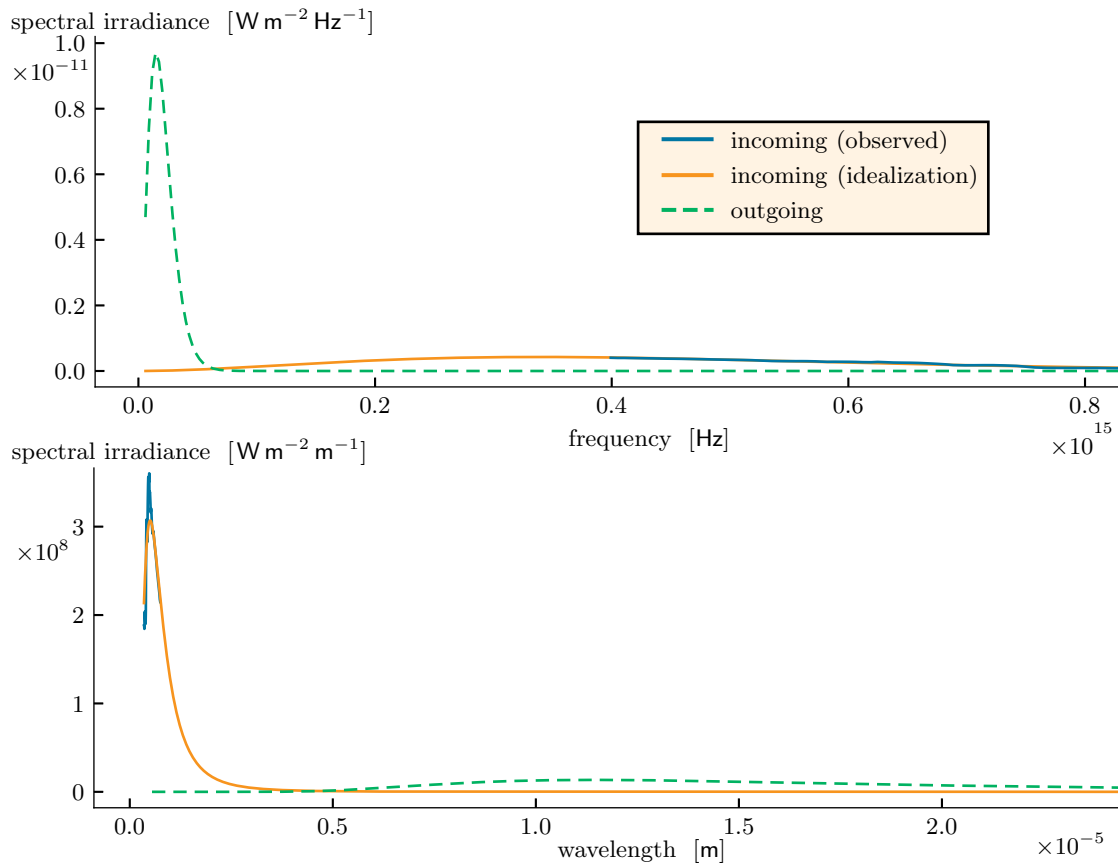


Figure 15.4: [Experimental data; mathematical functions.] **Earth without an atmosphere.** *Solid blue curves:* Observed spectral irradiance of light arriving at Earth from the Sun. The normalization of this curve includes a reduction for reflectivity (albedo), as well as averaging over surface position, time of day, and season. *Solid orange curves:* Planck spectral irradiance function at temperature 5760 K, also averaged and reduced by the geometrical and reflectivity factors in Equation 15.4. *Dashed green curves:* Planck spectral irradiance function with temperature $T_e = 254 \text{ K}$ chosen to match the total incoming irradiance (area under the orange curve); see Your Turn 15B. This curve has negligible overlap with the others. (a) Spectra in terms of frequency. (b) The same spectra in terms of wavelength. Problem 3.2 (page 142) works out how to convert between these two representations of a spectrum. [Experimental data from Thekaekara et al., 1969.]

In fact, all three of the planets we are considering arrive at steady-state temperatures for which they radiate most of their energy in the mid-infrared wavelength range ($\lambda = 10\text{--}20 \mu\text{m}$). In this range their emissivities are $\approx 100\%$.¹¹

Your Turn 15B

Use the last fact mentioned, Equation 15.4, and the Stefan–Boltzmann law to compute the predicted mean surface temperatures of naked versions of Earth, Venus, and Mars. Compare to the observed data in the preceding table.

Our laughably oversimplified approach has in fact done surprisingly well at predicting the mean surface temperature of Mars. However, the other two planets are

¹¹Perhaps surprisingly, even snow and ice have high IR absorptivity, and hence also high emissivity.

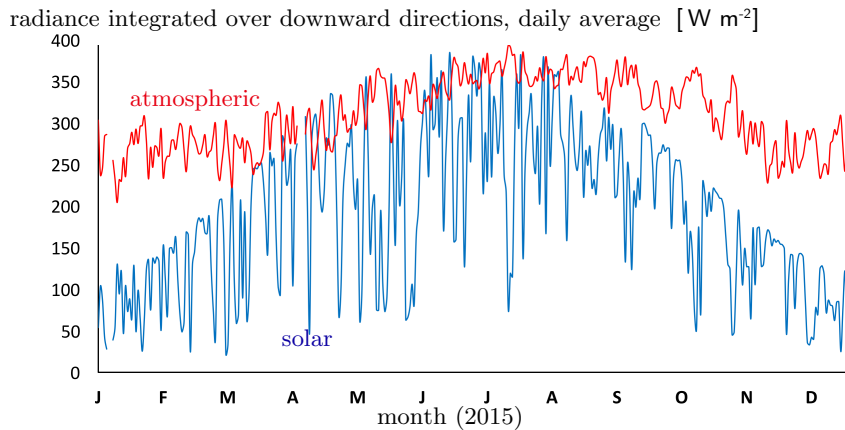


Figure 15.5: [Experimental data.] **Origins of infrared photons arriving at Earth’s surface.** The *lower (blue) curve* shows the spectral radiance integrated over all downward directions and over the band 300 to 2800 nm, which includes nearly all light coming directly from the Sun but excludes nearly all thermal radiation at Earth’s temperature (see Figure 15.4). The *upper (red) curve* shows the same quantity but instead integrated over the band 4500 to 42 000 nm, which excludes nearly all light coming directly from the Sun but includes nearly all thermal radiation at Earth’s temperature. [Data courtesy Peter Pilewski; see Pilewski, 2017.]

dismal failures: For Earth, our estimate of mean surface temperature is below the freezing point of water. Venus, with its higher reflectivity, was predicted to be even colder, and yet its actual surface temperature is hot enough to melt lead!

Your Turn 15C

Our Moon has radius 27% as great as Earth. It reflects about 14% of incoming sunlight. With no atmosphere or oceans to spread energy, and a slow rotation period, the “average” temperature is not very meaningful, but make a prediction anyway based on the same reasoning as above. (You may neglect the fact that the Moon spends some of its time shadowed by Earth.)

15.4 EARTH AND VENUS, DRESSED IN GRAY

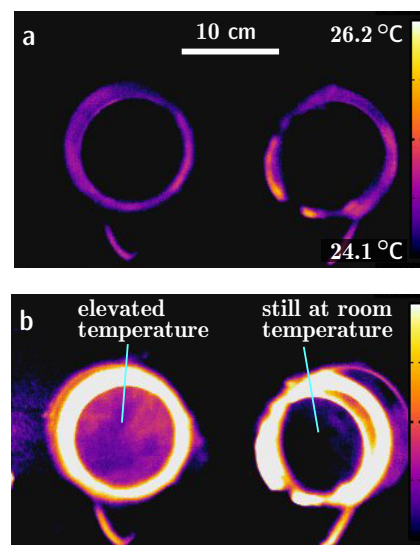
15.4.1 Effect of infrared-active gases

Earth and Venus are both “clothed” in mixtures of gases. But the dominant molecules (including N_2 , O_2 , and CO_2) are all weakly absorbing throughout much of the frequency range of the incoming solar photons. Hence, an atmosphere does not affect our preceding calculations very much.¹² So it may not be immediately obvious what is wrong with those calculations.

Figure 15.5 shows clearly where we have gone astray. The total energy input to Earth’s surface is actually dominated by a new source that we have not yet considered: Our atmosphere itself radiates infrared down to the surface. Although the dominant

¹²Water vapor does absorb some infrared radiation present in the solar spectrum. However, water vapor condenses at low temperatures, and so is confined to be near the surface; we will lump its absorption with that of the surface. Water *droplets* have a big effect on visible light, but mainly to scatter (and reflect) it; we already accounted for this reflection. Also, incoming ultraviolet light is mostly absorbed in Earth’s high atmosphere, but its total impact on the energy balance is small.

Figure 15.6: [Infrared photographs.] **Energy absorption by an IR-active gas.** Two identical, cylindrical chambers with transparent ends, viewed in the wavelength band $7.5\text{--}14\ \mu\text{m}$. False color indicates radiance in this band (reds are higher than blues); the scale bar is labeled with approximate inferred temperature values in degrees Celsius. The chamber on the right contains dry air. The one on the left contains carbon dioxide. Both have axial length 23 cm and are viewed end-on. (a) Both chambers started at room temperature. False color when looking into each chamber matches the backdrop. (b) The chambers were briefly exposed to infrared light. After irradiation was stopped, the one containing IR-active gas was observed to be slightly warmer for about one minute. That is, more infrared light was observed coming out of this chamber than was the case for either the backdrop or the other chamber.



molecular species in our atmosphere do not interact much with light in the visible range, some of them—notably H_2O and CO_2 —are infrared-active. They must be what is radiating. Figure 15.6 shows a lab experiment illustrating infrared activity of CO_2 gas.

It may seem paradoxical that energy from our atmosphere, whose ultimate origin is the Sun, should be arriving at a higher rate than solar energy itself! But consider again the vacuum flask mentioned earlier. Suppose that it contains an electric heating element that constantly delivers energy at some rate P . If the reflection from the walls were perfect, the energy input would raise the interior temperature without limit. Even with realistic, imperfectly reflecting walls, energy from the outgoing thermal photons will eventually be reflected back into the interior at a rate faster than P . There is no contradiction of energy conservation, because the hot interior also radiates energy toward the walls at an equally large rate (plus P).

In short, a planetary atmosphere can obstruct the loss of energy by absorbing some of what is emitted from the surface and emitting new infrared photons, some of which are directed back toward the planet (Figure 15.7b). Some planets, including Earth and Venus, have atmospheres that are “optically thick” at infrared wavelengths (that is, a photon emitted by the surface is unlikely to travel unimpeded out to space). In that situation, only the outermost region of that atmosphere will be able to radiate freely to space. That high layer’s temperature may be approximately given by an estimate like the one made in Section 15.3,¹³ but the planetary surface will in general be warmer.

The phenomenon just outlined is popularly called a “greenhouse” effect, but we will avoid that term. A greenhouse also admits visible photons, which warm its interior. But the purpose of the walls is mainly to prevent loss of energy to air that then moves away from the greenhouse, via convection or wind. If we remove the walls, those effects are what make the greenhouse (or any house) get cold in winter. In contrast, a planetary atmosphere impedes energy loss in the form of *photon emission*. A planet

¹³See Equation 15.7 (page 437) below.

Figure 15.7: [Cartoon.] **Energy budget of a planet.** (a) Solar radiation arrives at a naked planet (*left*) and warms the surface, which in turn radiates a larger number of lower energy photons (*right*). (b) In the presence of gas molecules (*circles*), incoming solar radiation is again almost unimpeded, but some of the outgoing photons are absorbed by infrared-active gas molecules, warming the atmosphere. Those molecules in turn radiate, in some cases back down towards the surface. The density of molecules decreases with height, so outgoing radiation can escape freely if it originates at high enough altitude, but the temperature at that altitude is lower than on the surface. Another energy-transport mechanism is available as well: The physical movement of molecules (convection), indicated by the *gray circulation* at right.

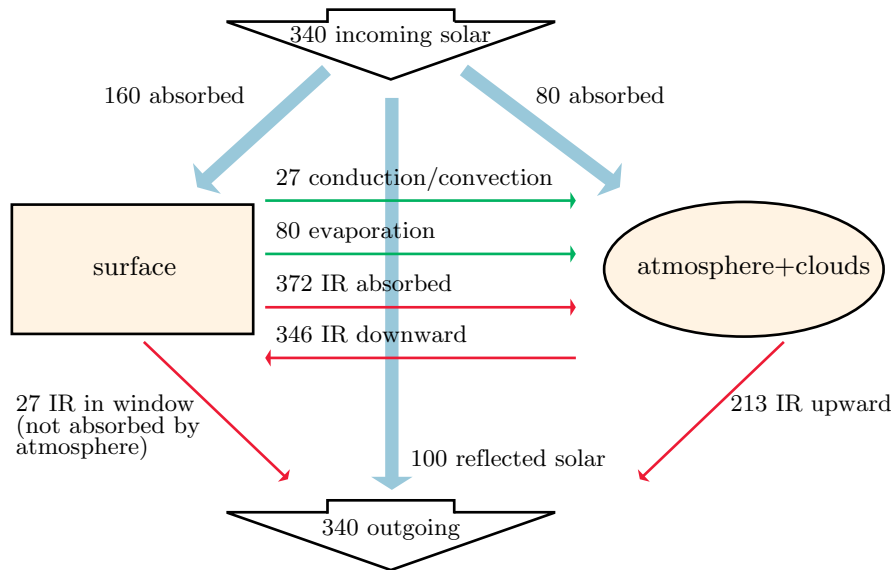
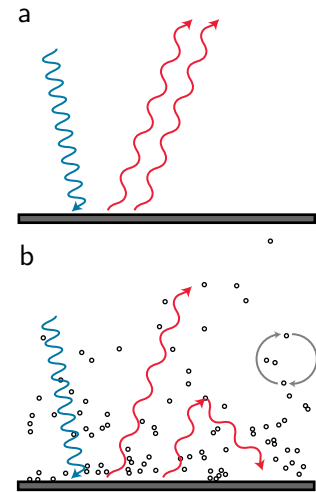


Figure 15.8: [Schematic.] **Energy transformations in Earth’s surface and atmosphere.** The global annual mean energy budget of the Earth system, simplified to just two compartments. All figures are in W m^{-2} . Short-wavelength photons are denoted by *blue arrows*; long-wavelength photons by *red arrows*. This chapter simplifies the picture by neglecting solar energy absorption by the atmosphere, and by lumping all lower-atmosphere transactions into the assumption of constant lapse rate. For example, the evaporation of water from the oceans and from the leaves of plants absorbs energy; the resulting water vapor then rises before giving back the energy as it condenses. Section 15.5.2 discusses the absorption “windows” that let some infrared radiation escape directly from the surface to space (*lower left*). [From Zagoni, 2015.]

with little atmosphere, such as Mars, is cold, but not because of convective cooling or wind loss! Instead, a naked planet’s surface must be at the temperature determined by the Stefan–Boltzmann law—considerably colder than that of a “clothed” planet.

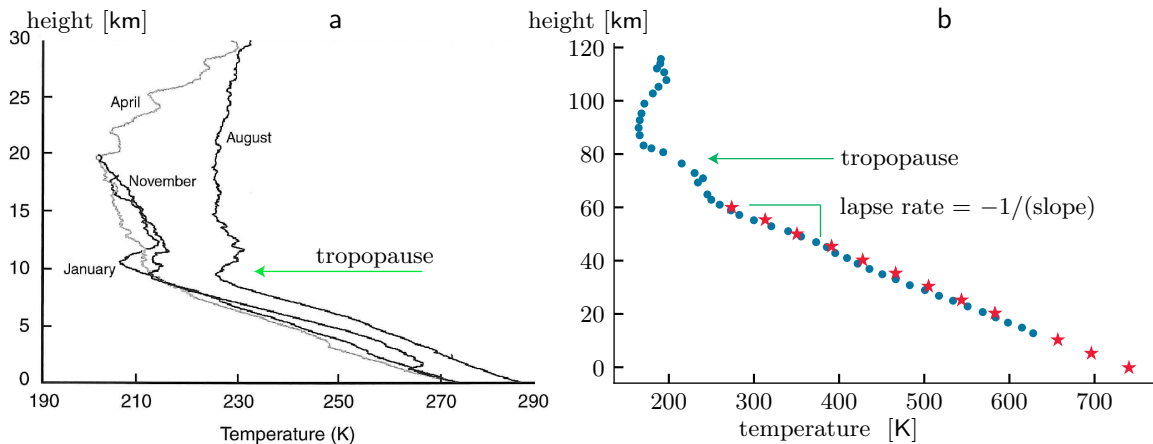


Figure 15.9: [Observational data.] **Atmospheric temperature profiles.** (a) Temperature profiles measured from balloons over Scandinavia, in various seasons. Summer is warmer than winter, but all seasons have a similar lapse rate. Beyond 30 km the temperature increases, but this has little relevance for climate, because most of the atmosphere’s mass lies in the region shown. (b) Temperature profile for Venus, derived from probes descending near the equator. The tropopause is at much higher altitude than on Earth. [Data from (a): Taylor, 2005 and (b): Seiff, 1983.]

Without infrared-active gases in its atmosphere, Earth would not be habitable.

Figure 15.8 shows the main flows of energy through the Earth system, and in particular the dominant role played by infrared radiation.

15.4.2 The lapse rate

We would like to improve our planetary energy flow calculation by incorporating the effects of infrared-active gases, at least qualitatively.

Figure 15.9 gives us an idea for how to start. Both on Earth and on Venus, the lowest, densest, part of the atmosphere shows a strikingly *linear* drop in temperature with altitude, despite large changes in density. The rate of decrease is called the **lapse rate**,¹⁴ and the zone over which it is roughly constant is called the **troposphere**.

If a planet’s atmosphere contains a significant concentration of infrared-active molecules, then its lower layers will be opaque to infrared light, and photon emission will not be the most effective way for energy to flow away from the surface (as it must). Solids and liquids can transfer thermal energy by conduction, but this process is not very effective in gases. Instead, air will physically move from warmer regions to cooler regions, because it is buoyant when heated from below. This process was called **convection** in Section 15.4.1.

Convection acts to reduce any temperature gradient, but only down to a certain point: Once the lapse rate decreases below a critical value, then an air layer will be stable against convection, even if a temperature gradient is present.¹⁵ We can estimate that critical lapse rate by a simple dimensional argument. For example, “dry air” (a gas roughly obeying the ideal gas law, without any component that can condense

¹⁴Beware: This quantity is not a “rate” in the sense of quantity per time. Rather, it is a spatial *gradient*.

¹⁵Rising warm air expands and hence cools, losing its buoyancy, so there is a minimal value for the gradient before convection turns on.

upon cooling) will have lapse rate $-g/c_p$, where g is the acceleration of gravity at the surface and c_p is the specific heat at constant pressure, per mass. This formula is nearly independent of gas density, because c_p is constant for an ideal gas.

A convecting region of atmosphere with the critical lapse rate is also said to follow the “dry adiabat.” The formula just given predicts its value as around 10 K/km.

Your Turn 15D

Work out the units in the formula to see how we arrive at K/mm. Then confirm the numerical estimate just given.

Similar but more accurate formulas can be written to account for an atmosphere with a condensing component, such as water vapor on Earth. The actual observed lapse rate in the troposphere is about 65 K/(10 km) (Figure 15.9a). For Venus, the breakpoint between the two regimes is less clear, but the data in Figure 15.9b again fall roughly on a straight line.

15.4.3 The tropopause and the outermost IR-thick layer

Now that we have a simplified picture of convection, let’s work upward from the planetary surface. Incoming photons in the visible wavelength range arrive there,¹⁶ and the fraction that is not reflected is absorbed by oceans and/or land. Section 15.4.2 said that in the troposphere, incoming energy is mainly transported upward by convection with fixed lapse rate. At some altitude, however, the remaining atmosphere is so thin that convection becomes ineffective and photon emission takes over. The critical altitude is called the **tropopause**; it is shown in Figure 15.9.

In more detail, recall that Section 9.4.2’a (page 311) introduced the absorption cross section a_1 as a measure of a molecule’s ability to absorb light of a given frequency. In this section, we will suppose that the cross section is independent of frequency throughout the infrared region, a simplification sometimes called the **gray gas model**.¹⁷ A uniform sample of thickness b absorbs a beam of light by the exponential law (Equation 9.6): That is, it removes the fraction $1 - e^{-a_1 cb}$, where c is the concentration. More generally, if the concentration is nonuniform we simply replace $a_1 cb$ by the **optical depth** $\tau = a_1 \int dz c(z)$. Note that despite the word “depth,” this quantity is dimensionless.

In a planetary atmosphere, the overall density of gas falls exponentially with altitude. The number density of a well-mixed chemical species such as CO₂ then also falls off exponentially: $c(z) = c_0 e^{-z/L}$, where c_0 is the density at the surface and L is called the **scale height** (Figure 15.10a). We are interested in the optical depth from a given z to infinity, because this quantity describes how much the atmosphere impedes the escape of photons:

$$\tau_{\text{escape}}(z) = a_1 \int_z^\infty dz' c_0 e^{-z'/L} = a_1 c_0 L e^{-z/L}. \quad (15.5)$$

We see that the optical depth to infinity is also exponential in form.

¹⁶As in Section 15.4.1, we continue to neglect any absorption of solar radiation by the atmosphere. Figure 15.8 shows that such absorption is actually significant.

¹⁷We continue to assume that absorption cross section is zero in the *visible* range. Section 15.5 will give a more detailed discussion.

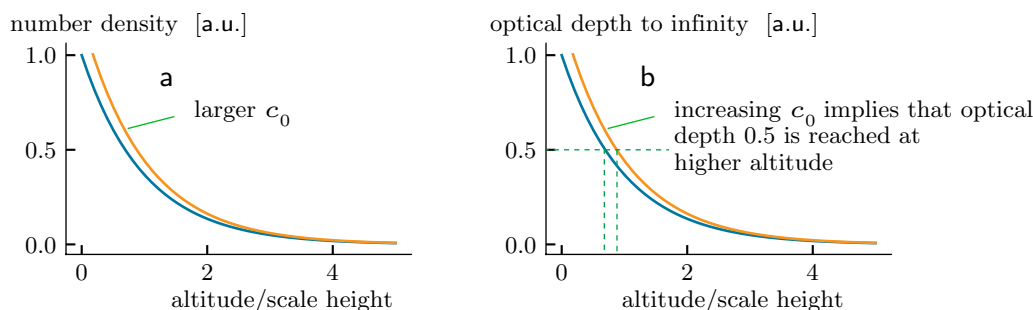


Figure 15.10: [Sketch graphs.] **Effects of changes to atmospheric composition.** (a) Number density $c(z)$ of a particular constituent molecule as a function of altitude z . Two different values of c_0 are shown. (b) Corresponding optical depths from z to infinity. A critical value such as 0.5 is achieved at different altitudes depending on the value of c_0 .

We now loosely define the “outermost IR-thick layer” of the atmosphere by the altitude z_{last} at which IR optical depth to infinity falls to some critical value, for example, 50%, and simplify by assuming that

- Everything below z_{last} transports energy outward solely by convection, with fixed lapse rate.
- Everything beyond z_{last} transports energy outward solely by photon emission, following the Stefan–Boltzmann law.

Thus, z_{last} corresponds roughly to the tropopause.

Although the assumptions just made are crude, they do incorporate the key insight of the preceding discussion:

For a clothed planet, the temperature estimate in Section 15.3 fails because it should have been applied to the outermost IR-thick layer, not to the surface. (15.6)

To do better than Section 15.3, we now suppose that the outermost IR-thick layer has temperature T_{last} and emissivity \mathcal{E} . Energy arrives at this layer from below in the form of infrared photons, at a total rate per area given by the net insolation (solar input, Equation 15.4). The layer absorbs a fraction \mathcal{E} of this energy, by Kirchhoff’s law. It emits infrared photons outward from its top, with total rate per area given by the Stefan–Boltzmann law: $\mathcal{E}\sigma_{\text{SB}}T_{\text{last}}^4$. But it also emits infrared *downward* from its lower boundary with the same irradiance. Energy balance then gives that

$$\mathcal{E}(1 - \mathcal{R})\sigma_{\text{SB}}T_{\text{sun}}^4 R_{\text{sun}}^2 / (4D_p^2) = 2\mathcal{E}\sigma_{\text{SB}}T_{\text{last}}^4. \quad (15.7)$$

Thus, T_{last} equals the naked-planet temperature from Section 15.3, times $2^{-1/4}$. Note that the value of \mathcal{E} is not needed, because it cancels from Equation 15.7.

For Earth, your result from Your Turn 15B implies that $T_{\text{last}} \approx 214 \text{ K}$, roughly as seen when we seasonally average the data at the tropopause (the breakpoint at 10 km height in Figure 15.9a). This simple calculation also predicts fairly well the temperature outside your window during a long-distance international flight.

For Venus, the left dot in Figure 15.9b is also roughly at the predicted $T_{\text{last}} \approx 190 \text{ K}$.¹⁸

¹⁸This value results when you multiply the value you obtained in Your Turn 15B by $2^{-1/4}$.

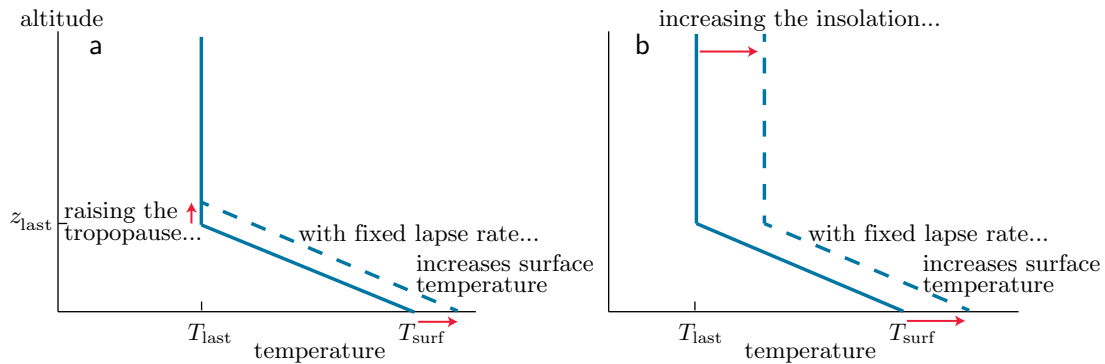


Figure 15.11: [Sketch graphs.] **Effects of changes to atmospheric composition or surface reflectivity on surface temperature.** (a) A change in atmospheric composition, with no change in reflectivity, can raise surface temperature by moving the tropopause upward (Figure 15.10b). (b) A decrease in reflectivity, with no change in composition, can raise surface temperature by increasing the net insolation and hence T_{last} .

15.4.4 Effects of changing atmospheric composition or albedo

Idea 15.6 gives a simple physical model:

- Temperature decreases linearly with increasing altitude as we ascend from the surface, at the critical lapse rate. The net outward flow of energy at each altitude always equals the net insolation, because in steady state energy must balance in each atmospheric layer.
- The temperature T_{last} at the outermost IR-thick layer is set by Equation 15.7.

Now consider those facts in the reverse order, that is, moving downward. We have found the temperature T_{last} at the tropopause, and we know how it changes on its way to the surface, *so we can predict the temperature at the surface* if we also know the altitude z_{last} of the tropopause.

Earth's high atmosphere contains trace quantities of carbon dioxide, methane, and other infrared-active gases. Increasing the concentration of those gases, for example, in the aftermath of a massive volcano eruption or as a result of industrial activity, raises the tropopause (Figure 15.10b) and hence warms the surface (Figure 15.11a).¹⁹ An increase of net insolation, for example, by a reduction of reflectivity after the loss of polar ice, also warms the surface by a different route (Figure 15.11b).

We also see the cause of the very high surface temperature on Venus: Its atmosphere is 96% carbon dioxide, a strongly infrared-active gas compared to Earth's oxygen and nitrogen. Moreover, Venus has much *more* atmosphere than Earth: Its surface pressure is 92 times as great as ours. For both of these reasons, its atmosphere remains opaque to infrared photons even up to high altitudes, leading to a much higher tropopause than on Earth (Figure 15.9b), and a correspondingly high surface temperature.

15.5 BEYOND THE GRAY GAS MODEL

¹⁹Changing the concentration of a noncondensing trace molecule, such as carbon dioxide in Earth's atmosphere, makes a negligible change in the specific heat, and hence in the critical lapse rate.

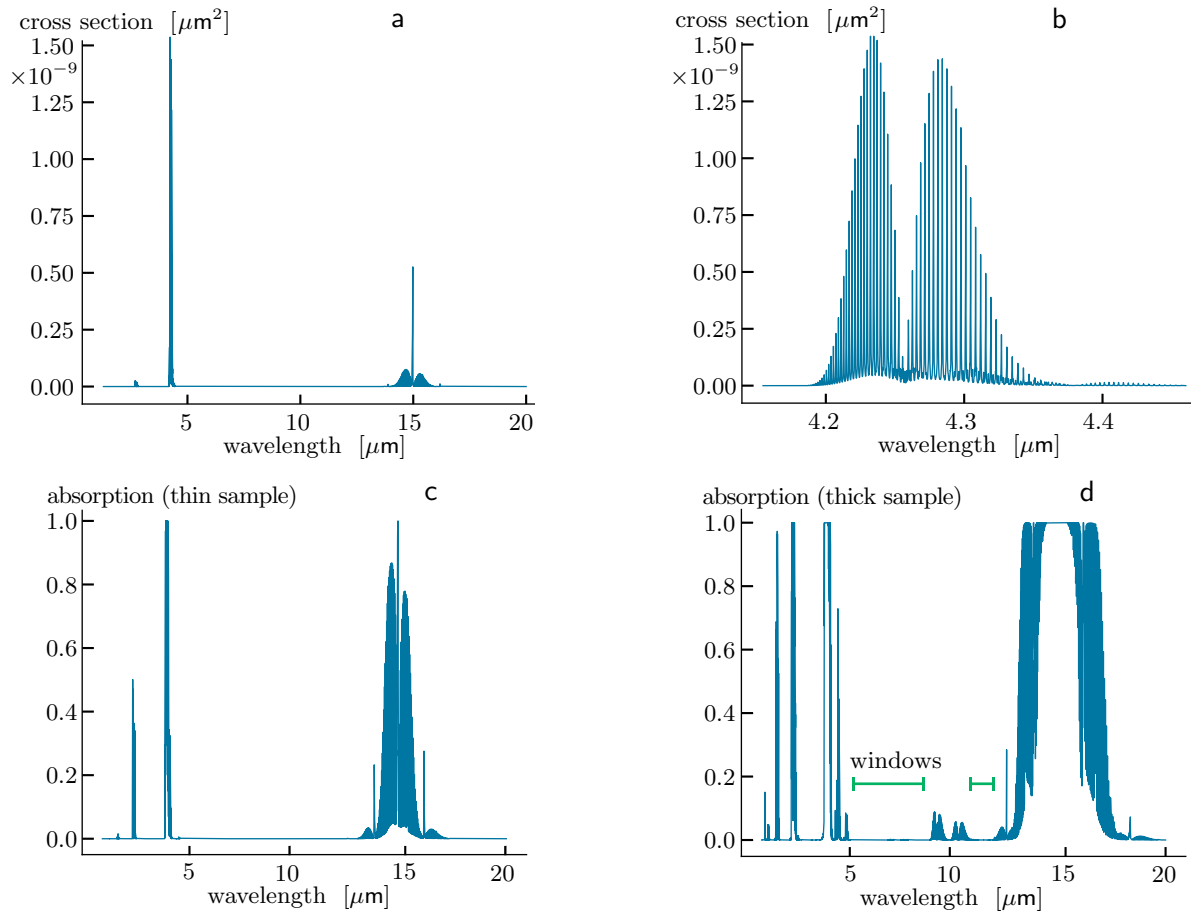


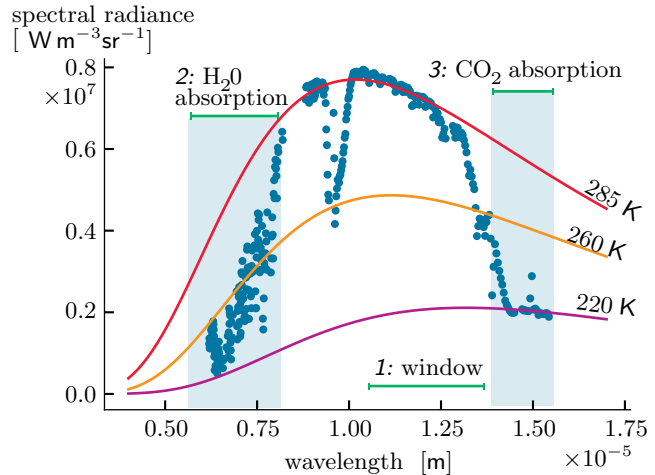
Figure 15.12: [Experimental data.] **Absorption spectrum of CO_2 .** (a) Absorption cross section at atmospheric pressure and room temperature, as a function of wavelength. (b) Detail of the largest peaks in (a). (c) Absorption of a sample of pure CO_2 with depth 0.01 m. The graph was obtained by applying Equation 9.7 (page 312) to the data in (a). (d) Absorption of a sample of pure CO_2 with depth 0.23 m, for example, the sample shown in Figure 15.5. For this thick sample, the absorption bands are wider than in (c): Features that are scarcely visible in (a) have large effects (see Section 15.5.2). The *green bars* show two “windows,” that is, bands of low absorption. [Data courtesy Eugene Clothiaux; see Bohren & Clothiaux, 2006, Fig. 2.12.]

15.5.1 Molecular IR absorption spectra are complex

Section 15.4.3 introduced the concept of the outermost IR-thick atmospheric layer by integrating absorption cross section down from infinity to altitude z (Equation 15.5) and defining z_{last} as the altitude where a particular threshold value is reached. Our discussion implicitly assumed that the cross section was constant (the “gray gas model”), but Figure 15.12 shows this to be far from true. Nor is this a peculiarity of carbon dioxide: All IR-active gases have complex absorption spectra.

Optical depth depends on the absorption cross section, and hence it, too will be different for different wavelengths. In particular, each wavelength has its own separate value for the altitude $z_{\text{last}}(\lambda)$ of its outermost IR-thick layer. The following sections will outline how this fact affects planetary climate calculations.

Figure 15.13: [Experimental data.] **Earth’s emission spectrum viewed from space.** The *dots* are observational data from the Atmospheric Infrared Sounder instrument; for comparison, the *curves* show thermal radiation spectra at various temperatures. **1:** In an IR-transparent window, energy streams out freely at the surface temperature. **2:** In a region where water vapor absorbs strongly but CO₂ does not, we see thermal radiation at a temperature corresponding to the one at which water vapor condenses out; higher than this, there is very little H₂O. **3:** In a region where CO₂ absorbs strongly but water vapor does not, we see thermal radiation at a still cooler temperature corresponding to the upper atmosphere, with a transition between this band and neighboring regions. [Data courtesy Yi Huang, from Huang et al., 2007.]



15.5.2 Absorption windows

We can roughly describe the message of Figure 15.12 by saying that carbon dioxide gas has absorption bands separated by broad transmission “windows.” Even when we add in the other relevant IR-active atmospheric constituents (mainly water vapor), we still find big windows in which the entire atmosphere is effectively transparent, for example, the range 10.0–12.5 μm . In this range, there is *no* value $z_{\text{last}}(\lambda)$ satisfying $\tau_{\text{escape}}(z_{\text{last}}, \lambda) = 0.5$; that is, the altitude of the “last” IR-thick layer is zero.

In an IR-transparent window, thermal radiation at Earth’s surface temperature streams freely away to space, unlike the scenario in Section 15.4.3. Moreover, because emissivity is equal to absorptivity (Kirchhoff’s relation), in such a window no additional IR radiation from the atmosphere joins the outgoing surface photons. An extraterrestrial observer thus sees thermal radiation at the surface temperature in these bands (region 1 in Figure 15.13).

Now imagine subdividing the absorption bands into wavelength slices and rank-ordering them in increasing order of cross section.

- For the transmission windows (bands with $z_{\text{last}}(\lambda) = 0$), the discussion in Section 15.3 is still valid. In these windows, the flux of energy is of Planck form,²⁰ at the surface temperature T_{surf} .
- The bands with lowest nonzero value of $z_{\text{last}}(\lambda)$ also radiate with the Planck spectrum, but with a temperature $T(z_{\text{last}}(\lambda))$ that is lower than T_{surf} by $z_{\text{last}}(\lambda)$ times the lapse rate (region 2 in Figure 15.13).
- The bands with the next higher value of $z_{\text{last}}(\lambda)$ are emitted from still cooler layers (region 3 of Figure 15.13), and so on.

The value of T_{surf} is now determined by the requirement that the sum of all the outward energy flows must balance the net insolation, just as with a naked planet (Section 15.3). As in the simpler gray gas model (Section 15.4), however, we again see that the effect of the atmosphere is to set a higher surface temperature than that of the corresponding naked planet.

²⁰See Equation 15.2 (page 426).

With this framework, we can now revisit the question of what happens when the concentration of an IR-active gas such as CO_2 is raised. For wavelengths deep within a strong absorption band, the story is similar to what was said in Section 15.4.4: $z_{\text{last}}(\lambda)$ increases, but the temperature at that altitude must stay fixed in order to radiate the required amount of energy, so the surface temperature must go up. Because this phenomenon does not occur in the transmission windows, however, we see that the change in surface temperature for a given change in CO_2 concentration (the **climate sensitivity** for CO_2) will be smaller than we would have expected in the gray gas model.

However, there is now an additional effect not present earlier. An absorption band will get *wider* as we increase the concentration of an IR-active component (Figure 15.12c,d). Thus, at the edges of a band (the “wing” regions) z_{last} will increase from zero to some nonzero altitude as the concentration rises high enough to create the first IR-thick layer. The wavelength zones that are able to impede outgoing IR radiation thus get wider, and so the quasisteady surface temperature rises faster than would have been predicted by the mechanism in the preceding paragraph acting alone.²¹

Climate sensitivity is further enhanced by *feedback*. For example, an increase in CO_2 will warm the surface by the two mechanisms just described. On Earth that warming, in turn, will drive more water vapor into the atmosphere from the oceans. Because water vapor is itself IR-active, its increase leads to still more warming. Conversely, the Earth system has in the past gotten stuck in a “snowball” state that was too cold to permit any appreciable water vapor content.

15.5.3 Pressure broadening and another look at Mars

Section 15.3 treated Mars as “naked,” and yet actually its atmospheric concentration of CO_2 is much larger than Earth’s! To resolve this contradiction, we need to appreciate another physically interesting aspect of molecular absorption.

An isolated molecule has well-defined quantum states, for example, rotationally and vibrationally excited states. Transitions between those states give off or absorb photons of sharply defined wavelength (Section 12.2.7). Those transitions are not *perfectly* sharp, however, because the lifetimes of the excited states are not infinite.²² And the lifetimes can become much shorter still if the molecule, instead of being isolated, is constantly colliding with others. As the total pressure of a gas sample increases, so does the density, and hence the time between collisions goes down. The corresponding increase in absorption line widths is called **pressure broadening**. The pressure relevant to this phenomenon is the total gas pressure, not just the part attributable to the IR-active gas, so it can be appreciable even though the concentration of CO_2 in Earth’s current atmosphere is very small.

Section 15.5.2 argued that the “wings” of a molecule’s absorption spectrum are critical for determining its surface-warming effect. The preceding paragraph argued that those wings are strongly influenced by ambient pressure. This observation helps explain why lots of CO_2 on Mars does not imply a large warming effect: Oxygen and nitrogen, the main constituents of Earth’s atmosphere, are largely absent on Mars, because its lower surface gravity cannot hold onto them as well as Earth’s does.

²¹The addition of gases not naturally present in the atmosphere, such as chlorofluorocarbons, can also create entirely new absorption bands in what was previously a window region.

²²This is another manifestation of the quantum uncertainty relation mentioned in Section 4.6.3’ (page 174).

Without pressure broadening from these gases, Mars's CO_2 is less effective at blocking IR radiation than on Earth. Less warming from CO_2 (as well as greater distance to the Sun) means a cooler planet less able to evaporate water; moreover, the H_2O molecule is even lighter than N_2 or O_2 , another reason why there is so little of it in the atmosphere. All of these reasons combine to render Mars nearly as cold as if it had no atmosphere at all, as you saw in Your Turn 15B.

15.5.4 A road not (yet) taken on Earth

Conversely, Venus's very high atmospheric pressure makes its CO_2 even *more* effective than on Earth. Why is Venus's atmosphere so different from ours? Earth and Venus have similar composition and move in similar orbits, so it's reasonable to suppose that they were similar in other respects when they first formed. On Earth, carbon continually cycles between the atmosphere and the planet's interior, providing a stabilizing feedback, but part of this loop seems to have broken on Venus.

One clue is that *Venus no longer has any oceans*. In light of that fact, a possible scenario runs as follows. At some point, the temperature on Venus rose higher than Earth's. That rise evaporated more water from the existing oceans. Water is infrared-active, so the extra vapor drove surface temperature still higher, in a "runaway" (positive feedback loop). Eventually *all* surface water was in vapor form. Liquid water is needed for the physical process of weathering rocks, which on Earth slowly removes atmospheric CO_2 and ultimately returns it to the planet's interior, so losing liquid water breaks the carbon cycle.²³ Meanwhile, volcanic activity continued unabated, continually adding CO_2 from the interior to the atmosphere.

Gradually, solar UV photons broke up atmospheric water vapor and the light hydrogen escaped to space. The remaining free oxygen may in turn have been removed from the atmosphere, for example, by oxidizing crustal elements such as iron.²⁴ The rising carbon dioxide level took over the role of impeding infrared energy loss from the atmospheric water vapor.

The foregoing scenario is based on real phenomena known to operate on Earth, but it is still speculative. What is not in dispute is that however Venus arrived at its current atmosphere, it is now stuck in a permanently hot state.

THE BIG PICTURE

Planetary climates are determined in part by an intricate chain of energy *transformations*: Incoming light, primarily in the visible range, converts its energy to thermal motion of Earth's surface materials. That energy exits back into space by the net effect of convection and atmospheric absorption and emission, ultimately as photons in a wavelength band different from the one that dominated the incoming light. Details of the matter/light interaction, which is inherently quantum mechanical in character, determine the different effects of various gases, with huge implications for life.

²³On Earth, photosynthesis by living plants also removes CO_2 from air: Photosynthetic marine organisms eventually fall to the bottom of the ocean, carrying their fixed carbon into sediments that ultimately reenter Earth's crust, another part of the carbon cycle that relies on liquid water (and mild temperatures).

²⁴Earth, too, had very little free oxygen prior to the rise of photosynthetic organisms.

FURTHER READING

Semipopular:

Climate: Ingersoll, 2013.

Intermediate:

Planetary climate: Jaffe & Taylor, 2018; Pierrehumbert, 2011; Wilson & Gea-Banacloche, 2012; Taylor, 2005.

Sensitivity to CO₂ changes: Andrews, 2010.

Technical:

Planetary climate: Pierrehumbert, 2010.

Light and the atmosphere: Stamnes et al., 2017.

Infrared thermography: Vollmer & Möllmann, 2018.

PROBLEMS

15.1 *Planck spectrum*

First work Problem 3.2.

- We can interpret the Planck spectrum as a probability density function for photon frequency: Just divide Equation 1.19 by the energy per photon and normalize it. Call the result \wp_ν . Use a computer to find the value of ν where it is maximal (this depends on temperature).
- For an arbitrary PDF, we cannot assume that the location of the maximum is the same as the expectation of ν , so compute that expectation and compare to your answer in (a).
- Transform the distribution into a PDF for the wavelength, $\lambda = c/\nu$. Then find the value of λ where \wp_λ is maximal.
- Also find the expectation of λ and compare to the results of (a–c).
- However, the *median* frequency will correspond to the *median* wavelength in the expected way, and so gives an objective answer to the question “What is a typical photon?” Compute the value ν_m with the property that any photon has probability 1/2 to have frequency below ν_m .

15.2 *Scaling of thermal radiation*

An ordinary incandescent light bulb gives off light with a spectrum that is close to the thermal form. That is, the density function for energy to arrive at a detector exposed to this light, per frequency interval, is a constant times Planck’s formula, Equation 1.19. (The constant depends on the detector’s size, distance to the source, and so on.) In some communities it is more traditional to express such spectra in terms of energy per wavelength interval, so use your result from Problem 3.2 to convert the thermal radiation formula to this form.

Obtain Dataset 17,²⁵ which contains experimental data on the energy streaming out of a light bulb at two different temperatures. Those temperatures were not measured directly; however, the total power consumption of the bulb was measured in each case to be 73 W and 55 W, respectively. A consequence of Equation 1.19 is that total power radiated from a hot body to space as thermal radiation equals a constant

²⁵<http://www.physics.upenn.edu/biophys/PtN/Datasets/17Planck/index.html>.

times the body's temperature raised to the fourth power. Assume that other energy losses are negligible.

Only a part of each spectrum was measured, corresponding to visible light. Explore the hypothesis that this object emits thermal radiation by seeking two values of absolute temperature, related by the ratio $(73/55)^{1/4}$, whose Planck spectra match the given data up to an overall constant common to both cases.

15.3 Planck spectrum via simulation

Section 1.3.3'c (page 52) gave a simple derivation of the Planck spectrum, but one step was valid only in the limit of a large cavity: We supposed that we could treat each possible momentum \mathbf{p} separately, then integrate the corresponding contributions to energy density over $d^3\mathbf{r}d^3\mathbf{p}/(2\pi\hbar)^3$. In this problem you'll do a more exact derivation, which also retains the statistical character of light instead of just computing an average energy density.

Consider a box shaped as a cube with length L . Idealize the box walls as perfect conductors, so that the only photons allowed are those that come from light modes with wavenumber of the form $\mathbf{k} = (\pi/L)(\eta_x, \eta_y, \eta_z)$, where η_i are three positive integers (Section 13.3). The energy of one photon is then $\hbar c\|\mathbf{k}\|$.

- Get a computer to make a list of all triples (η_x, η_y, η_z) satisfying $\|\boldsymbol{\eta}\| < 30$. For concreteness, consider a box with $L = 14\ \mu\text{m}$ containing photons in equilibrium at $T = 5000\ \text{K}$. Duplicate each triple to account for the two polarization states a photon may have (Chapter 13).
- Idea 1.20 implies that for each frequency, direction, and polarization, the number of photons is a random variable with Geometric distribution. All of these distributions are statistically independent of each other because photons do not interact among themselves. Get your computer to draw an appropriately distributed random integer for each allowed photon state, and hence populate the states.
- The frequency of a photon in this box is a discrete variable. Nevertheless, we may subdivide a range of frequencies into equally spaced bins, count up all the photons from (b) that appear in each bin, and create a bar chart of the results that resembles a histogram. So get your computer to do that. (For a large box the frequency values would become very dense, so in that limiting case your answer becomes a PDF.)
- Multiply Equation 1.19 by L^3 and divide by $2\pi\hbar\nu$ to convert it into a density of photons per frequency interval, then compare to your result in (c).